MATHEMATICAL OLYMPIAD SUMMER PROGRAM 1999

GEOMETRIC INEQUALITIES

ZVEZDELINA STANKOVA-FRENKEL
UC BERKELEY

Note: For $\triangle ABC$ we denote by α, β, γ, a, b, c, h_a, h_b, h_c, l_a, l_b, l_c, m_a, m_b, m_c, r, R, r_a, r_b, r_c and S its angles, sides, altitudes, angle bisectors, medians, inradius, circumradius, exradii and area.

1. Prove that
$$\sum(2a-p)(b-c)^2 \geq 0$$
with equality iff $\triangle ABC$ is equilateral. Note that the inequality is equivalent to any of the following two:
$$3(a^3+b^3+c^3+3abc) \leq 4p(a^2+b^2+c^2);$$
$$p^2 \geq 16Rr-5r^2.$$

2. Let the angle bisectors in $\triangle ABC$ intersect the opposite sides in points D, E, and F, and let S' be the area of $\triangle DEF$.
   (a) Prove that $\dfrac{3abc}{4(a^3+b^3+c^3)} \leq \dfrac{S'}{S} \leq \dfrac{1}{4}$.
   (b) If $a=5$ and $\dfrac{3abc}{4(a^3+b^3+c^3)} = \dfrac{5}{24}$, find b and c, given that they are integers.

3. Let O be the circumcenter of acute $\triangle ABC$. Lines AO, BO and CO intersect BC, CA and AC in points A_1, B_1 and C_1. Prove that $OA_1+OB_1+OC_1 \geq 3R/2$.

4. Let $A_0A_1...A_n$ (n–even) be an $(n+1)$–gon with circumcenter O and circumradius R. Lines OA_i intersect the opposite sides of the polygon in points B_i. Prove that
$$OB_0+OB_1+\cdots+OB_n \geq \frac{n+1}{n}R,$$
with equality iff O is the centroid of the polygon.

5. Prove that for an arbitrary triangle:
$$a^2+b^2+c^2 \leq \frac{72R^4}{9R^2-4r^2}.$$
When is equality attained?

6. Prove that for an arbitrary triangle:
   (a) $\prod(b+c) \leq 8pR(R+2r)$;
   (b) $\sum bc(b+c) \leq 8pR(R+r)$;
   (c) $\sum a^3 \leq 8p(R^2-r^2)$.
7. For an acute $\triangle ABC$ prove that
$$2abc\left(abc+p\sum a^2-\sum a^3\right) \geq 5\prod(b^2+c^2-a^2).$$
8. Prove the inequalities:
   (a) $\prod(p-a) \geq \prod(2a-p)$;
   (b) $\sqrt{3}\sum \frac{bc}{l_a} \geq 4p$;
   (c) $R\sqrt{3}\sum \frac{h_a}{l_a} \geq 2p$;
   (d) $\sqrt{3}\sum \cos\frac{\beta-\gamma}{2} \geq 2\sum \sin\alpha$;
   (e) $4\left(\sqrt{3}\prod \cos\frac{\beta-\gamma}{4} - 2\prod \cos\frac{\beta}{2}\right) \geq \sqrt{3}$;
   (f) $3\sqrt{3}\frac{\sum \frac{bc}{l_a}}{\sum bl_b} \geq 4\sqrt{\frac{2p}{abc}}$.
9. Let the triangle with sides equal to the medians m_a, m_b and m_c have inradius r_m and circumradius R_m.
   (a) Prove or disprove:
$$r_m \leq \frac{3abc}{4(a^2+b^2+c^2)},$$
   where equality is obtained iff the original $\triangle ABC$ is equilateral.
   (b) Prove that
$$R_m \geq \frac{a^2-b^2+c^2}{4p}.$$
10. For an arbitrary triangle show the inequalities:
   (a) $m_a m_b m_c \geq r(m_a^2+m_b^2+m_c^2)$;
   (b) $12Rm_a m_b m_c \geq \sum b(c+a)m_b^2$;
   (c) $4R\sum bm_b \geq \sum bc(b+c)$;
   (d) $2R\sum \frac{1}{bc} \geq \sum \frac{m_b}{m_c m_a}$.